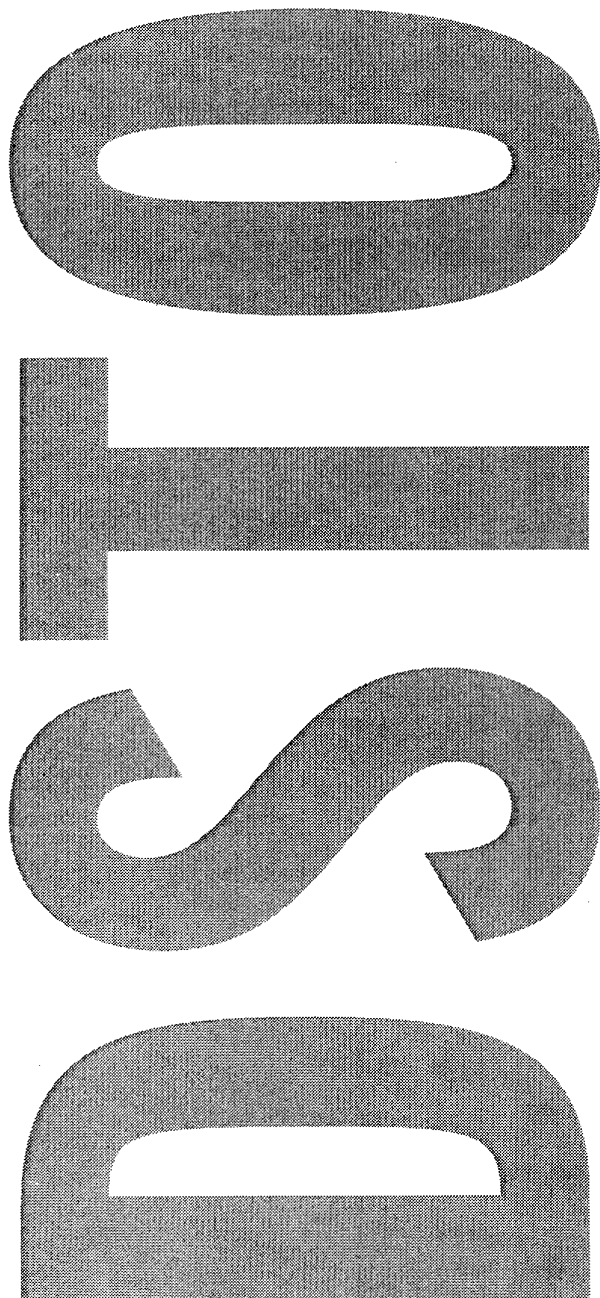




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**On The Character and  
Complexity of Certain Defensive  
Resource Allocation Problems**

W.P. Malcolm

DSTO-TR-1570

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## On The Character and Complexity of Certain Defensive Resource Allocation Problems

*W. P. Malcolm*

**Weapons Systems Division**  
**Systems Sciences Laboratory**

DSTO-TR-1570

### **ABSTRACT**

In this article we consider two classes of static defensive resource allocation problems, these are, the static "target-value based" weapon target allocation and the static "asset-value based" weapon allocation problem. It is shown that the target-value based problem can be recast, (using indicator functions), into an instantiation of the so-called transportation problem. The transportation problem can be solved by numerous polynomial-time algorithms and has received considerable attention in the literature. We also consider the so-called "asset-based" weapon target allocation problem. This problem is shown to be somewhat more difficult than the target value based problem. A simulation study is presented for the target-value allocation problem, with emphasis upon sensitivity to uncertain target-elimination probabilities.

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# On The Character and Complexity of Certain Defensive Resource Allocation Problems

## Executive Summary

The primary aims of this report are to introduce and explain, certain technical issues concerning defensive resource allocation problems. However, the core aim is to revisit the so-called weapon target allocation problem and consider its implications in the modern context of a networked defence. It is clear that the foremost issue arising from a networked defence, in our context, is that the number and diversity of available defensive resources will significantly increase. It is therefore timely to consider both the character and complexity of defensive resource algorithms. Quite apart from any particular context, optimal defensive resource allocation has two main classes of applications, these are, 1) the online scenario, that is, committing defensive resources in real time, during real engagements and 2) the offline scenario, that is, using allocation algorithms to simulate and model the effectiveness of defensive resources against a given threat scenario. The importance of the online scenario is immediate, however, the offline scenario also has significant value and can perceivably be used to aide acquisition, or to estimate a measure of preparedness. Further, a capability to consider offline scenarios will most likely enhance the development of online algorithms. This claim follows naturally from the inherent complexity in defensive resource allocation problems, which often necessitate unavoidable approximation for online applications.

In this report we begin with a literature survey, starting from approximately 1950. Various models with various objectives are discussed. We also consider the diversity of approaches taken to solve defensive resource allocation problems.

For a particular example, we consider the so-called static target-value based problem. A special case of this problem is shown to be amenable to a linear programming formulation and can be readily solved with the simplex algorithm. This algorithm is a standard algorithm in linear programming and is used to solve certain constrained optimisation problems. The special case we consider is indeed useful, as it provides a convenient means of studying weapon target allocation and potentially gaining insight in to more complex scenarios. Further, this special case is unique, in that despite being cast as a relaxed linear program, with decision variables ranging in the interval  $[0,1]$ , its natural solutions are guaranteed to take integer values in the set  $\{0,1\}$ . Consequently, sensitivity analysis is also possible using this formulation.

A computer simulation is provided, showing the cost of uncertainty in target-elimination probabilities. To make this report as self contained as possible, a proof of the fundamental Theorem of linear programming is provided in the Appendix.

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# 1 Introduction

A current trend, if not revolution in military affairs, is to recast traditional platform-centric notions of defence, into a form of networked defence [1,26,27]. This shift in thinking has many far reaching implications. It is certainly true, that the emerging questions concerning implementation of networked defence are highly nontrivial. Typically, these questions of implementation have concerned mainstream areas in Electrical Engineering, such as Communications, RADAR, Tracking and Fusion etc. For example, how does one manage defence information on a complex and often heterogeneous asynchronous time-varying network ?

Currently there is much emphasis placed upon surveillance and situation awareness aspects of networked defence, usually with the objective of constructing a battle-space picture, however, a battle-space picture alone offers *no defence* ! In this report we turn our attention to the *action* of defence, in particular, the optimal allocation of defensive resources. Put briefly, we consider questions such as how best to commit defensive resources when in a situation of imminent and real threat. A distinction we make is as follows; we are not primarily concerned with the above technical issues pertaining to a network, rather, we think of ourselves as consumers of what a network provides and subsequently act on this information with the aim to best deploy defensive resources. It is worth noting, that here the word *best* could have varied meanings, such as preferentially defending a collection of assets, or minimising the survival of a threat.

The optimal allocation of defensive resources against imminent threats has a long and varied history, however, the rough beginning of this research may be taken as the 1950s. Indeed, in the 1950s and 1960s, the Operations Research Journal published many papers on defensive resource allocation. Some of this foundation literature appears to have been motivated by the then Cold War threat of Intercontinental Ballistic Missile exchange between the former Soviet Union and the United States of America. This perceived threat is an enduring one and arguably one of the most controversial in the history of defence science. One indication of levels of concern over missile defence, is that in the United States alone, this problem received more defence funding than any other defence research program in history.

Considering the current trends towards network centric ideas in defence, it is indeed timely to revisit the area of optimal defensive resource allocation. A networked defence potentially augments the number and variety of available defensive resources and indeed changes the very nature of how defensive resources might be committed. To introduce some foundation in defensive resource allocation, this report begins with an overview of the general problem area and provides a literature survey identifying significant contributions. We restrict our attention to two main static model classes, these are, the target-value based allocation problem and the asset-value based allocation problem. It is shown, that the target-value based problem can be recast as a particular instantiation of the well known transportation problem and subsequently solved by a linear programming algorithm. A basic problem that arises in this recourse to linear programming, is our decision variables are necessarily integer-valued, typically binary, where 0 might denote not committing a resource and 1 denotes committing a resource. Put simply, if we relax this requirement, we allow continuous range decisions in  $[0, 1]$ . Immediately this approximation manifests

a serious problem. Suppose a solution is computed to commit 0.765 of a resource. To address such problems, one might make use of either of the two main integer programming techniques, cutting plane methods ([78]) or branch and bound [48]. However, the target-value problem admits a useful special case when the number of targets is equal to the number of defensive resources and each target must be addressed. In this case the matrix in the constraint set is unimodular. What this means, is that all basic solutions are naturally integer-valued. The simplex algorithm can be used to solve this problem. The value in considering this special case, is it provides a convenient means of gaining insight in to weapon target allocation problems, which otherwise are very difficult to solve. Indeed, there is only one exact solution to the static weapon target allocation problem due to den Broeder et al [22]. Note however, Den Broeder's solution considers only the very restrictive case of a single weapon class.

This article is organised as follows. In section 2 we present a brief literature survey, covering the main contributions to defensive resource allocation problems and the diversity of methods proposed to obtain solutions. In section 3 we concentrate on some specific resource allocation problems, in particular the so called target-value based weapon target allocation problem. This particular problem is explained in some detail and is shown to be amenable to linear programming, under certain assumptions. We also discuss the so called asset-based weapon target allocation problem in this section. To conclude section 3, we consider the core issue of algorithm complexity arising from resource allocation problems, all of which are at least non-deterministic-polynomial-time hard. In section 4 we consider the issue of robustness of target-value based algorithms. A computer simulation is given, showing the consequences of uncertainty in target-elimination probabilities. In section 5 we give a conclusion, placing emphasis on the customer relevance (ADF), of defensive resource allocation. Finally, in section 6, we propose future research in defensive resource allocation.

## 2 Literature Survey

Defensive resource allocation is a vast area, with numerous particular scenarios and numerous techniques for computing solutions. It is impossible to comprehensively survey this work in a brief technical report, so we first give essentially a guide to the literature, identifying key elements *etc.* We then identify a particular set of modelling techniques which have been frequently proposed to solve allocation problems.

### 2.1 General Overview

As is often the case with a significant body of literature, one can usually identify a small set of key contributions, upon which many subsequent contributions were based. Most of the early contributions to defensive resource allocation appeared in the journal *Operations Research* (1950s, 1960s). This collection of literature is almost entirely restricted to the static class of problems. In defensive resource allocation, there are two general classes of models studied, these are, 1) the so called *static* problems and 2) *dynamic* problems, where exchanges are modelled over sequential stages in time. For a static problem, one

essentially considers a defence *action* as a single exchange. These problems embody no explicit notion of time, that is, one measures a threat, one devises a response and one commits that response. Although time does not appear explicitly in static problems, the time required to compute solutions is of course a primary issue. This class of problems holds an important place in defensive resource allocation, as under many common circumstances, the sequential assignment of resources against targets might just not be an option. Further, while static problems might seem simplistic, or incomplete, they are highly relevant. Studying these problems often enhances the development of their temporal counterparts. Another important division in the literature is the distinction between *offensive* resource allocation and *defensive* resource allocation. It is interesting to note that early foundation literature consists mainly of the offensive scenario, whereas most of the more recent literature is cast for the defensive scenario. One also finds literature simultaneously considering both scenarios, this is typical for game theoretic analyses of resource allocation. This is sometimes called the attack-defence game.

A suitable starting point for the static problem literature is arguably the paper by Alan Manne [51]. Manne's paper further developed a model proposed by Merrel Flood and suggested some properties of the model. The model discussed in Manne's article is a deterministic static weapon target allocation model which has since been considered by many different approaches.

Three substantial surveys of the contributions to weapon target allocation problems have appeared in the literature, these are, 1) Matlin [53], 2) Eckler and Burr [23] and 3) Mettler and Preston [56]. These reports present extensive, yet different in character surveys. In Matlin's article, the emphasis is upon optimal allocation for the purpose of offence. He collects a number of articles and gives a brief discussion of them, he also proposes a classification scheme of algorithms for defensive resource allocation. The report due to Eckler and Burr is substantial and detailed. Its emphasis is upon defence, rather than offence. This report is not limited to allocation problems, but considers other related technical issues, such as models for fractional damage etc. The report by Mettler and Preston is restricted to the asset-based defence problem, a varied set of algorithms for asset-based defence are described in some detail. A feature common to all three of these surveys is that defensive resource allocation is restricted to static scenarios.

## 2.2 Mathematical Models and Techniques

In most of the problems concerning defensive resource allocation, one is confronted with constrained optimisation problems. Loosely, optimisation has two broad classes, calculus based techniques and enumerative based techniques. Naturally the calculus based techniques rely on a classical notion of a gradient (with some exceptions). However, all defensive resource allocation problems admit only integer-valued solutions. This means that to use calculus based methods, one must first relax the primary problem, assuming this approximation does not render the subsequent solution completely meaningless with respect to the original problem. Enumerative solutions impose no restrictions such as differentiability or linearity *etc.*

### 1. Dynamic Programming

Dynamic programming methods have been applied to the defensive resource allocation

tion [4,16]. In this report we consider a special case of the target-value based weapon target allocation problem, which can be cast as a linear programming problem. This class of problem is closely related to the so called  $\{0,1\}$ -Knapsack problem, which can also be solved by dynamic programming, see [76]. While the dynamic programming method offers some scope for resource allocation problems, it quickly become unsuitable for a large number of states, or stages in the sequential case. This issue is loosely referred to as Bellman's curse of dimensionality. Dynamic programming is discussed in [3,9,10,13]. A recent historical article is given in [21]

## 2. Lagrange Multiplier Methods

Lagrange multiplier methods have been applied to the defensive resource allocation [24,67]. In particular, an extensive account of Lagrange Multiplier methods applied to weapon target allocation is given in the research monograph [19]. Lagrangian relaxation is also considered in the Thesis of Hosein [40], however, these methods make use of derivatives, which will not always exist for many resource allocation problems. Hence, a key problem here, is that Lagrange multiplier methods may not always lead to the true optimal solution.

## 3. Game Theory

Game Theoretic methods have been applied to defensive resource allocation problems in [18]. See also the recent article on missile defence [63]. Quite distinct from the mainstream of defensive resource allocation, game theory considers the allocation objectives from both sides simultaneously, assuming a two-player scenario as is typical for warfare. A core problem with game theory, in this setting, is one assumes knowledge of strategy spaces for both competing sides, which in military operations is rarely the case. Two alternatives to this issue are, 1) develop a scheme robust to uncertainty (ie risk sensitive or robust estimation), or 2) strive to learn the strategy space of an adversary during the process of the game. Seminal contributions in game theory are [43,80], see also [14].

## 4. Integer Programming

Integer programming provides a natural framework for  $\{0,1\}$  allocation problems. However, most of the standard integer programming methods are deterministic. Recently stochastic methods have been considered in the article [60]. One important drawback with integer programming, is that it is generally known not to be robust. Further, in most cases it is not possible to apply sensitivity analysis. Excellent treatments of integer programming are given in [32,78]

## 5. Neural Networks

Neural Network methods have been applied to defensive resource allocation problems in [77]. Neural Networks have a long history, which some authors date from the McCulloch and Pitts, 1943, [54]. This early work was motivated a need to model computational processes, as they are believed to happen in the human brain. It was understood that processing time was not the core issue, rather, the organisation of processing. Neural Networks have enjoyed many successful applications, particularly in pattern classification and pattern recognition. However, computational time, or training time, remains a problematic issue with Neural Networks. This limitation makes this approach unsuitable for defensive resource allocation. A reasonable review of Neural Networks can be found in [17].

## 6. Genetic Algorithms

Genetic algorithms have been applied to defensive resource allocation problems in [31, 59]. The optimisation procedure referred to as a Genetic Algorithm was introduced by Holland in 1975 [38]. Genetic algorithms are guided random searches and in this sense are related to simulated annealing [34, 44, 69]. The random nature of the mechanism of genetic algorithms makes it difficult in general to analyse their performance. One impressive feature of genetic algorithms, is their ability to escape local optima. An excellent survey of genetic algorithms is given in the article [75].

## 7. Neuro-Dynamic Programming

Neuro-dynamic programming (NDP) has been applied to defensive resource allocation problems in the recent article [6]. Neuro-dynamic programming (also referred to as reinforcement learning), proposes an approximation to dynamic programming by approximating the reward function. A key issue, is precisely *how* to approximate this function. Comprehensive treatments of NDP are given in [8, 74], see also the review/summary articles [12].

## 2.3 Summary

It is clear that the literature in defensive resource allocation has developed unevenly. The foundation static models, offensive or defensive, target-value based or asset-value based, represent the beginnings of this literature, starting with Manne's article and followed by the articles listed and reviewed in [23, 53, 56]. Following this foundation, the next most significant contribution was the superb PhD Thesis of Hosein. Hosein's Thesis marks, in some sense, the introduction of a time parameter and therefore, the beginning of *dynamic* resource allocation problems. Numerous important theoretical results were established in this Thesis. Separate from Hosein's Thesis there have been numerous less standard approaches to weapon target allocation, such as genetic algorithms, simulated annealing and neural networks. Some recent literature, see [6, 60, 63, 77], would suggest that research in defensive resource allocation still remains a *very* active area.

Finally, the reader should be aware that research in defensive resource allocation, such as the US Missile Defence program, is in most cases highly classified research and so is not reported in the open literature. It is also interesting to note that the foundation literature in the journal Operations Research was drawn substantially from companies supporting US defence, such as: The General Electric Company, The Rand Corporation, The Lambda Corporation and the Mitre Corporation, to name just a few. It is perhaps not surprising then, that the literature in defensive resource allocation might well be described as *scant*.

## 3 Static Model Classes

In this article we consider two model classes for static weapon target allocation. In the first class we consider the so called target-value based models. In these models one attributes a numerical value to attacking targets and it is ultimately the ranking (threat evaluation), of this value, combined with the defensive resource target-elimination probabilities which determine the optimal defence.

In the second model class, we consider the so called asset-value based models. These models represent what is usually referred to as "preferential defence", that is, one chooses to defend a collection of assets each ranked according to a given value. These models are far more demanding to solve than target-value based models, however, the motivation to consider them is primarily that defence effectiveness can be substantially improved when the information required by this approach, (situation awareness), is available. It is clear that this class of model has a natural application in a networked defence.

### 3.1 Target-Value Based Deterministic Allocation

All probabilities are defined on the fixed space  $(\Omega, \mathcal{F}, P)$ .

#### 3.1.1 The Standard Model

Suppose the *Defence*, located at some fixed point, has  $1, 2, \dots, M$  defensive resources (possibly weapons of some description). Suppose an *Attacker* has fired  $1, 2, \dots, N$  weapons, each aimed directly at the point location of the *Defence*. In the sequel we refer to the attackers weapons as targets. We denote by  $A_{(i,j)}$ , the event that the defensive resource  $j \in \{1, 2, \dots, M\}$  eliminates target  $i \in \{1, 2, \dots, N\}$ . Write

$$p_{(i,j)} \triangleq P(A_{(i,j)}) = P(\text{Target } i \text{ is eliminated by resource } j). \quad (1)$$

We refer to the probabilities  $p_{(i,j)}$  as target-elimination probabilities. These probabilities are usually assumed as known.

What we would like, is to construct an objective function representing the average survival of the collective threat, or the target set  $1, 2, \dots, N$ , given a particular allocation of defensive resources, and define this average survival in terms of the elimination-probabilities of the available responses. To this end, we will need to consider target-elimination probabilities of allocations and as is standard in the literature. The following independence is assumed

$$\begin{aligned} f_i &= P((\Omega \setminus A_{(i,1)}) \cap (\Omega \setminus A_{(i,2)}) \cap \dots \cap (\Omega \setminus A_{(i,M)})), \\ &= P(\Omega \setminus A_{(i,1)}) P(\Omega \setminus A_{(i,2)}) \dots P(\Omega \setminus A_{(i,M)}). \end{aligned} \quad (2)$$

To label any particular allocation, we introduce binary decision variables and a corresponding action space.

Write

$$x_{(i,j)} \triangleq \begin{cases} 1 & \text{Allocate Resource } j \text{ to Target } i, \\ 0 & \text{Ignore Target } i. \end{cases} \quad (3)$$

Given we consider  $N$  targets and  $M$  defensive resources, the action space for this problem is the binary space

$$\mathbb{B} = \{0, 1\} \times \{0, 1\} \times \dots \times \{0, 1\} = \{0, 1\}^{N \times M}. \quad (4)$$

Further, a given response, or feasible solution, can be thought of as a matrix in the form:

$$\begin{bmatrix} x_{(1,1)} & x_{(1,2)} & x_{(1,3)} & \dots & x_{(1,M)} \\ \vdots & & & & \vdots \\ x_{(N,1)} & x_{(N,2)} & x_{(N,3)} & \dots & x_{(N,M)} \end{bmatrix}. \quad (5)$$

Each of the attackers targets are assigned a value  $v_i \in (0, 1]$ . These parameters are assumed given by some form of threat evaluation, for example

$$v_i \triangleq f(\text{estimated time of arrival of target } i), \quad (6)$$

$$v_i \triangleq g(\text{size/yield of target } i), \quad (7)$$

$$v_i \triangleq h(\text{range of target } i). \quad (8)$$

**REMARK 1** The scalar quantities  $v_i$  cannot be predicted in advance and are therefore completely random. It is difficult however to propose a suitable distribution for these numbers, so they are modelled simply as parameters, whose value is determined at the time of threat.

Finally, to construct an objective function, we consider the so called leakage probabilities,

$$\prod_{j=1}^M (1 - p_{(i,j)})^{x_{(i,j)}} = (1 - p_{(i,1)})^{x_{(i,1)}} \times (1 - p_{(i,2)})^{x_{(i,2)}} \times \dots \times (1 - p_{(i,M)})^{x_{(i,M)}}. \quad (9)$$

This probability represents the survival of target  $i$ , given the shown allocation. It is now clear that our objective should have the form:

$$F \triangleq \sum_{i=1}^N v_i \left\{ \prod_{j=1}^M (1 - p_{(i,j)})^{x_{(i,j)}} \right\}. \quad (10)$$

Here  $F : \mathbb{B} \rightarrow \mathbb{R}$ . Further, it is clear that  $F$  is bounded below by zero and is bounded above by  $\sum_{i=1}^N v_i$ .

**REMARK 2** To interpret the function at (10), it is perhaps useful to represent it in the form

$$F = \sum_{i=1}^N v_i \Phi_i \left( [x_{(i,j)}]_{\substack{1 \leq i \leq N \\ 1 \leq j \leq M}} ; [p_{(i,j)}]_{\substack{1 \leq i \leq N \\ 1 \leq j \leq M}} \right). \quad (11)$$

The scalar quantity  $F$ , is a particular average survivability of the collective threat  $\{v_1, v_2, \dots, v_N\}$ , given a *particular* allocation  $\mathbf{x} = [x_{(i,j)}]_{\substack{1 \leq i \leq N \\ 1 \leq j \leq M}}$ .

The standard constraint used in the minimisation of  $F$ , is

$$\Psi(j) = \sum_{i=1}^N x_{(i,j)} = 1, \quad j \in \{1, 2, \dots, M\}. \quad (12)$$

This constraint imposes two demands, firstly, it ensures that each of the  $M$  resources is committed and secondly, it ensures that each resource can be allocated only once.

In summary, the constrained optimisation we must solve is:

Compute

$$F^* \triangleq \min_{\mathbf{x} \in \mathbb{B}} \{F\} = \min_{\mathbf{x} \in \mathbb{B}} \left\{ \sum_{i=1}^N v_i \left\{ \prod_{j=1}^M (1 - p_{(i,j)})^{x_{(i,j)}} \right\} \right\} \quad (13)$$

Subject to

$$\Psi(j) = \sum_{i=1}^N x_{(i,j)} = 1, \quad j = 1, 2, \dots, M. \quad (14)$$

**REMARK 3** The constrained optimisation problem defined by equations (13) and (14) presents some immediate difficulties. Firstly, the number of feasible solutions is clearly  $N^M$ . This complexity is indeed an issue in networked defence, as the number of available resources ( $M$ ), is most likely to be augmented by a network. For example, suppose  $N = 5$  and  $M = 10$ , then the number of feasible solutions is  $9765625 \approx 10 \times 10^6$ . A second and more problematic property of this problem, is the objective function  $F$ , with domain  $\mathbb{B}$ , is clearly not convex. It is also clear that the function  $F$  is nonlinear.

An inescapable feature of defensive resource allocation problems, as is evident in target-valued allocation problem just described, is that they are inherently integer-valued, that is, it makes no sense to commit any fraction of a defensive resource. A common approach to address this difficulty is relaxation, here one allows the decision variables  $x_{(i,j)} \in \{0, 1\}$  to assume a continuous range in the compact set  $[0, 1]$ . To consider this approach, we first check the convexity of the relaxed form of the function  $F$ , the motivation to do so is fundamental to optimisation, for example,

“the great watershed in optimisation isn’t between linearity and nonlinearity, but convexity and non-convexity”

R. Rockafellar, SIAM Review 1993.

Establishing the convexity of  $F$ , on  $[0, 1]^{N \times M}$ , will at least ensure the existence of an optimal solution.

**LEMMA 1 (CONVEXITY OF THE RELAXED OBJECTIVE FUNCTION)** *Suppose that each of the  $N \times M$  decision variables  $x_{(i,j)}$  are not integer-valued, rather, suppose these variables each take values in the compact set  $[0, 1]$ . Then the function  $F : [0, 1]^{N \times M} \rightarrow \mathbb{B}$ , defined at (13) is a convex function.*

#### Proof of Lemma 1

The objective function  $F$ , defined at (13), is a sum of  $i = 1, 2, \dots, N$  functions. Consider a sum of convex functions  $g_1, g_2, \dots, g_k$ , where for each  $g : \mathbb{R}^n \rightarrow \mathbb{R}$ , then, for two vectors  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  and  $\lambda \in [0, 1]$ . We note that

$$\sum_{j=1}^k g_j(\lambda \mathbf{x} + (1 - \lambda) \mathbf{y}) \leq \lambda \sum_{j=1}^k g_j(\mathbf{x}) + (1 - \lambda) \sum_{j=1}^k g_j(\mathbf{y}). \quad (15)$$



Since a sum of convex functions is again convex, we need only check convexity for an  $i$ -th function in the sum defining  $F$ . Without loss of generality we set  $v_i = 1$  and write

$$g_i(\mathbf{x}) \triangleq \prod_{j=1}^M (1 - p_{i,j})^{x_{i,j}}. \quad (16)$$

Suppose  $\alpha, \beta \in \mathbb{R}^n$ . We define a functional  $\phi : [0, 1]^M \rightarrow \mathbb{R}$ ,

$$\phi \triangleq \frac{g(\beta)}{g(\alpha)}. \quad (17)$$

Since  $g$  is strictly positive then  $\phi$  is well defined. Further, the functional  $\phi$  is convex, to see this note that

$$\phi^\lambda \leq \lambda \phi + (1 - \lambda), \quad \lambda \in [0, 1]. \quad (18)$$

Note that

$$\lambda g(\beta) + (1 - \lambda)g(\alpha) = g(\alpha)(\lambda \phi + (1 - \lambda)). \quad (19)$$

Similarly,

$$\begin{aligned} g(\lambda\beta + (1 - \lambda)\alpha) &= \prod_{j=1}^M (1 - p_{i,j})^{\lambda\beta_j + (1 - \lambda)\alpha_j} \\ &= \prod_{j=1}^M (1 - p_{i,j})^{\lambda\beta_j} (1 - p_{i,j})^{\alpha_j} (1 - p_{i,j})^{-\lambda\alpha_j} \\ &= \prod_{j=1}^M (1 - p_{i,j})^{\alpha_j} \left\{ \frac{\prod_{j=1}^M (1 - p_{i,j})^{\beta_j}}{\prod_{j=1}^M (1 - p_{i,j})^{\alpha_j}} \right\}^\lambda \\ &= g(\alpha) \phi^\lambda. \end{aligned} \quad (20)$$

Using the calculation immediately above and the convexity of  $\phi$ , we see that

$$\begin{aligned} g(\lambda\beta + (1 - \lambda)\alpha) &= g(\alpha) \phi^\lambda \\ &\leq g(\alpha)(\lambda \phi + (1 - \lambda)) \\ &\leq \lambda g(\alpha) \frac{g(\beta)}{g(\alpha)} + (1 - \lambda)g(\alpha) \\ &\leq \lambda g(\beta) + (1 - \lambda)g(\alpha). \end{aligned} \quad (21)$$

□

**REMARK 4** Minimising the relaxed form of the function given at (10) is a convex programming problem, that is, optimising a convex function, over a convex set.

### 3.1.2 A Uniform Defensive Resource Class Solution

One vivid indicator of the complexity of defensive resource allocation, is that there are very few exact solutions. However, one special case which admits an exact solution is worth noting.

Suppose one is facing  $n$  threats and one has only a uniform class of defensive resources, whose target-elimination probabilities are  $p_1, p_2, \dots, p_n$ . Further, suppose this scenario is made yet simpler, by assuming that  $p_1 = p_2 = \dots = p_n = p$ . It is also assumed that the target-values are nondecreasing and that one wishes to commit  $M$  available defensive resources, where  $m_i$  denotes the number of resources committed to target  $i$ , with  $\sum_{i=1}^n m_i = M$ .

Write

$$P_k \triangleq P(\text{at least } k \text{ targets are eliminated}). \quad (22)$$

**THEOREM 1** (DEN BROEDER, ELLISON, EMERLING [22]) *The probability  $P_k$ , of eliminating at least  $k$  threats, where  $k \in \{1, 2, \dots, n\}$ , is at its maximum if the elements of the set  $\{m_j\}_{1 \leq j \leq n}$  differ at most by unity.*

#### Proof of Theorem 1

To prove this Theorem, den Broeder *et al*, considered the quantity  $P_k$  under two different allocation policies and found that the difference in the resulting  $P_k$  values, showed that the maximum  $P_k$  is attained by applying the allocation policy stated in Theorem 1.

Suppose we divide the set of targets  $\{1, 2, \dots, n\}$  into two subsets  $A$  and  $B$ , where this division places two targets in  $A$  and the remaining  $n - 2$  targets in  $B$ .

Write

$$A \triangleq \{s, t\}, \quad s, t \in \{1, 2, \dots, n\} \quad (23)$$

$$B \triangleq \{1, 2, \dots, n\} \setminus A. \quad (24)$$

Suppose the integer-valued random variables  $x$  and  $y$ , denote, respectively, the number of target destroyed in the sets  $A$  and  $B$ .

Write

$$z \triangleq x + y. \quad (25)$$

Then

$$\begin{aligned} P_k &\triangleq P(\omega \mid z(\omega) \geq k) \\ &= P(\omega \mid y(\omega) \geq k)P(\omega \mid x(\omega) = 0) \\ &\quad + P(\omega \mid y(\omega) \geq k - 1)P(\omega \mid x(\omega) = 1) \\ &\quad + P(\omega \mid y(\omega) \geq k - 2)P(\omega \mid x(\omega) = 2). \end{aligned} \quad (26)$$

Consider a particular assignment  $\{m_1, m_2, \dots, m_n\}$ , from a total of  $m$  resources. Further, suppose target  $s$  is allocated  $m_s$  resources and target  $t$  is allocated  $m_t$  resources. Without

loss of generality, it is assumed that  $m_s > m_t$ . For the assignment just described, we note that

$$\begin{aligned} P(\omega | x(\omega) = 0) &= q^{m_s+m_t} \\ P(\omega | x(\omega) = 1) &= q^{m_s}(1 - q^{m_t}) + q^{m_t}(1 - q^{m_s}) \\ P(\omega | x(\omega) = 2) &= (1 - q^{m_s})(1 - q^{m_t}). \end{aligned} \quad (27)$$

Now we consider a second assignment, where  $m_s$  and  $m_t$  are replaced, respectively, by  $m_s - 1$  and  $m_t + 1$ . In a slight abuse of notation, we denote the  $P_k$  values resulting from these allocations by, respectively,  $P_k|_{m_s, m_t}$  and  $P_k|_{(m_s-1), (m_t+1)}$ . Computing the difference in these probabilities we get

$$\begin{aligned} P_k|_{m_s, m_t} - P_k|_{(m_s-1), (m_t+1)} &= P(\omega | y(\omega) \geq k)(q^{m_s+m_t} - q^{(m_s-1)+(m_t+1)}) \\ &\quad + P(\omega | y(\omega) \geq k-1)p(q^{m_t} - q^{m_s-1}) \\ &\quad - P(\omega | y(\omega) \geq k-2)p(q^{m_t} - q^{m_s-1}) \\ &= p(q^{m_t} - q^{m_s-1}) \times \\ &\quad (P(\omega | y(\omega) \geq k-1) - P(\omega | y(\omega) \geq k-2)) \\ &= p(q^{m_t} - q^{m_s-1})P(\omega | y(\omega) = k-2). \end{aligned} \quad (28)$$

Recall, it was assumed without loss of generality, that  $m_s > m_t$ . This means  $q^{m_t} - q^{m_s-1} \geq 0$ , so  $P_k|_{m_s, m_t} - P_k|_{(m_s-1), (m_t+1)} \leq 0$ . The implication of this observation, is that decreasing the difference  $|m_t - m_s|$  does not decrease  $P_k$ , but may in fact increase it. So, to ensure one computes the required maximum, viz  $\max_{m_1, m_2, \dots, m_n} \{P_k\}$ , it follows that  $|m_s - m_t|$  must be minimised. The consequences of this result prove the Theorem.  $\square$

**REMARK 5** Although the scenario studied by den Broeder *et al* is a simple one, who's outcome might well have been guessed, it is indeed a useful result. In the Thesis of Hoesein [40], this same scenario was considered and an alternative local search algorithm was proposed. The allocation algorithm arising from Theorem 1, due to den Broeder *et al*, is known in the literature as the Maximum Marginal Return Algorithm.

### 3.1.3 A Linear Allocation Model

In this section we consider a special case of the problems given by (13) and (10).

The form of the objective function given at (13) can always be rewritten by noting the equality

$$(1 - p_{(i,j)})^{x_{(i,j)}} = 1 - x_{(i,j)}p_{(i,j)}. \quad (29)$$

Further, suppose one considers an additional constraint, where  $M = N$  and no target is left unattended, then

$$\Gamma(i) = \sum_{j=1}^M x_{(i,j)} = 1, \quad j = 1, 2, \dots, M. \quad (30)$$

It follows that

$$\begin{aligned}
 \prod_{j=1}^M (1 - p_{(i,j)})^{x_{(i,j)}} &= (1 - p_{(i,1)})^{x_{(i,1)}} \times (1 - p_{(i,2)})^{x_{(i,2)}} \times \dots (1 - p_{(i,M)})^{x_{(i,M)}} \\
 &= 1 - x_{i,1}p_{(i,1)} - x_{(i,2)}p_{(i,2)} - \dots - x_{(i,M)}p_{(i,M)} \\
 &= 1 - \sum_{j=1}^M x_{(i,j)}p_{(i,j)}.
 \end{aligned} \tag{31}$$

Using the equality at (31), we get,

$$\begin{aligned}
 \min(F) &= \min_{\mathbf{x} \in \mathbb{B}} \left\{ \sum_{i=1}^N v_i \left\{ \prod_{j=1}^M (1 - p_{(i,j)})^{x_{(i,j)}} \right\} \right\} \\
 &= \min_{\mathbf{x} \in \mathbb{B}} \left\{ \sum_{i=1}^N v_i - \sum_{i=1}^N v_i \left\{ \sum_{j=1}^M x_{(i,j)}p_{(i,j)} \right\} \right\} \\
 &= \max_{\mathbf{x} \in \mathbb{B}} \left\{ \sum_{i=1}^N v_i \left\{ \sum_{j=1}^M x_{(i,j)}p_{(i,j)} \right\} \right\} \\
 &= \min_{\mathbf{x} \in \mathbb{B}} \left\{ -1 \times \sum_{i=1}^N v_i \left\{ \sum_{j=1}^M x_{(i,j)}p_{(i,j)} \right\} \right\}.
 \end{aligned} \tag{32}$$

Now the constrained optimisation problem reads:

$$\text{minimise} \quad F = -1 \times \sum_{i=1}^N v_i \left\{ \sum_{j=1}^M x_{(i,j)}p_{(i,j)} \right\} \tag{33}$$

$$\text{subject to} \quad \Psi(j) = \sum_{i=1}^N x_{(i,j)} = 1, \tag{34}$$

$$\Gamma(i) = \sum_{j=1}^M x_{(i,j)} = 1. \tag{35}$$

Note, this problem construction implies that  $M = N$ .

**REMARK 6** Imposing the two constraints at (34) and (35), avoids the unsettling scenario of leaving imminent threats unattended. For example, the problem defined by (13) and (34) admits a candidate feasible solution, where all defensive resources may be committed to just one target/threat.

**REMARK 7** The optimisation problem defined by (33), (34) and (35), is an instantiation of the so called transportation problem. This problem is well known and has received considerable attention in the literature. A feature of the transportation problem, is the basis matrices have a special structure which can be exploited. The transportation problem and other related network flow problems are discussed in [5, 25, 50, 70].

**REMARK 8** The constrained optimisation problem described above, admits  $M! = N!$  feasible solutions.

### 3.1.4 Linear Integer Programming Formulation

What we wish to do is to cast the problem defined at (33), (34) and (35), in the so called *standard form* for a linear program [50]. To this end, write

$$\tilde{\mathbf{x}} \triangleq (\mathbf{x}_{(1,1)}, \dots, \mathbf{x}_{(N,1)}, \mathbf{x}_{(1,2)}, \dots, \mathbf{x}_{(N,2)}, \dots, \mathbf{x}_{(1,M)}, \dots, \mathbf{x}_{(N,M)})' \in \mathbb{R}^{(N \times M) \times 1}, \quad (36)$$

$$\mathbf{c} \triangleq (-v_1 p_{(1,1)}, \dots, -v_1 p_{(1,M)}, \dots, -v_N p_{(N,1)}, \dots, -v_N p_{(N,M)})' \in \mathbb{R}^{(N \times M) \times 1}, \quad (37)$$

$$\mathbf{b} \triangleq (1, 1, \dots, 1)' \in \mathbb{R}^{M \times 1}, \quad (38)$$

$$A \triangleq \begin{bmatrix} 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & \dots & 1 \\ \vdots & & & & & & \ddots & & & & & & \vdots \\ 0 & 0 & \dots & & & & & & 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & \dots & 0 & 1 & 0 & \dots & & & & & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 1 & \dots & & & & & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 1 & \dots & & & & \dots & 0 \\ \vdots & & \ddots & & & & & & \ddots & & & & \vdots \\ 0 & 0 & \dots & 1 & & & & & & & & & 1 \end{bmatrix} \in \mathbb{R}^{(M+N) \times (M \times N)}. \quad (39)$$

The constrained optimisation problem now reads:

$$\text{Minimise} \quad f = \langle \mathbf{c}, \tilde{\mathbf{x}} \rangle \quad (40)$$

$$\text{Subject to} \quad A \tilde{\mathbf{x}} = \mathbf{b}, \tilde{\mathbf{x}} \geq \mathbf{0}. \quad (41)$$

**DEFINITION 1 (UNIMODULAR MATRICES)** A matrix  $A$  is said to be *totally unimodular* if the determinant of every square matrix formed from it has value  $-1$ ,  $0$ , or  $1$ .

(More detail on unimodular matrices can be found in the text [2].)

It is clear that the matrix  $A$  has rank  $M$ , further, it is also totally unimodular. In many linear integer programming scenarios, one can consider a relaxed form of an objective function and thereafter apply standard techniques, such as branch and bound or cutting plane methods to compute an integer-valued solution. However, in the scenario described here, one need only compute the standard linear programming solution. All vertices on the convex polytope for this problem correspond to naturally integer-valued basic solutions.

**PROPOSITION 1** The solution to the linear program given at (40) and (41), is naturally an integer-valued solution, taking a value in the binary space  $\mathbb{B} = \{0, 1\}^M$ .

To prove Proposition 1, we will need the following Lemma.

**LEMMA 2** An integer matrix  $A = [a_{(i,j)}]_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ , with all  $a_{(i,j)} \in \{-1, 0, 1\}$ , is totally unimodular if:

1. no more than two nonzero elements appear in each column,
2. the rows of  $A$  can be partitioned into two subsets  $M_1$  and  $M_2$  such that: a) if a column contains two nonzero elements with the same sign, one element is in each of the subsets, b) if a column contains two nonzero elements of opposite signs, both elements are in the same subset.

The following proof, by induction, is now standard and can be found in integer programming texts, for example [78]. See also [35, 36, 39, 81].

**Proof of Lemma 2**

One element sub-matrix of  $A$  has a determinant equal to  $(-1, 0, 1)$ . Suppose that the Theorem is true for all sub-matrices of  $A$ , being of order at least  $k - 1$ . Let  $B$  be any sub-matrix of  $A$  of order  $k$ . If  $B$  contains a null vector then  $\det B = 0$ . If  $B$  contains a column with only one nonzero element, we expand  $\det B$  by that column and apply the induction hypothesis. Finally, consider the case in which every column of  $B$  contains two nonzero elements. then from 2(a) and 2(b), for every column  $j$

$$\sum_{i \in M_1} b_{(i,j)} = \sum_{i \in M_2} b_{(i,j)}, \quad j = 1, 2, \dots, k \quad (42)$$

Finally, denote the  $i$ -th row of  $A$  by  $b_i$ , then the equality (42) gives

$$\sum_{i \in M_1} b_i - \sum_{i \in M_2} b_i = 0. \quad (43)$$

This clearly implies that  $\det B = 0$ .

□

**Proof of Proposition 1**

Referring to the Appendix, in particular, Definition 3 and Theorem 2 and from this section Lemma 2, we see that an optimal solution to the problem stated at equations 33, (34) and (35), if it exists, may be written as

$$\mathbf{x}_B^* = B^{-1} \mathbf{b} = \frac{B^+}{\det(B)} \mathbf{b}. \quad (44)$$

Here  $B^+$  denotes the adjoint of the matrix  $B$ . If  $B$  is unimodular, then it is also an integer-valued, then so is  $B^+$ , since the adjoint is constructed by the cofactors of the matrix  $B$ . Further, Lemma 2 ensures that,  $\det(B) \in \{-1, 0, 1\}$ . It follows that  $\mathbf{x}_B^*$  is purely integer-valued.

□

## 3.2 Asset-based Deterministic Allocation

Rather than choosing the target values alone to *drive* an allocation algorithm, one can sometimes construct a *preferential* defence strategy, where the optimisation is, in a sense, driven by ranked numeric values assigned to assets one wishes to defend. Naturally

asset-based defence applies to scenarios where the elements of a collection of assets are differently located, but each under threat of some form. One example might be three ships at sea. Suppose one of these ships is a command and control centre and is rated as the most-valued asset, then defensive resources are allocated against the collective threat to preferentially optimise its survival. A critical element in asset-based defence is the availability of situation awareness information. Put simply, we must be able to determine which targets are headed for which assets. In some literature this is known as the clustering problem, this problem could also be thought of as a data association problem.

### 3.2.1 The Standard Model

The objective functions arising in asset-based defence are somewhat different to those arising in target-value based defence. The following notation will be used.

#### DEFINITION 2

$$K \triangleq \text{The number of assets to defend} \quad (45)$$

$$N \triangleq \text{The number of targets} \quad (46)$$

$$M \triangleq \text{The number of defensive resource} \quad (47)$$

$$G_k \triangleq \text{Subset of targets aimed at asset } k = 1, 2, \dots, K, \#(\cup G_k) = N \quad (48)$$

$$n_k \triangleq \text{number of targets aimed at asset } k = 1, 2, 3, \dots, K \quad (49)$$

$$W_k \triangleq \text{the value of asset } k = 1, 2, \dots, K \quad (50)$$

$$\pi_i \triangleq \text{probability that target } i = 1, 2, \dots, N \text{ destroys the asset it's aimed at.} \quad (51)$$

$$p_{(i,j)} \triangleq \text{probability that resource } j \text{ eliminates target } i. \quad (52)$$

As before we define binary decision variables by:

$$x_{(i,j)} \triangleq \begin{cases} 1 & \text{Allocate Resource } j \text{ to Target } i, \\ 0 & \text{Ignore Target } i. \end{cases} \quad (53)$$

The critical factors in asset-based formulations are the estimated sets  $\hat{G}_k$ . These sets will rarely be known exactly and so are written  $\hat{G}$ , to stress this uncertainty. In defence parlance, the collection  $\{\hat{G}_1, \hat{G}_2, \dots, \hat{G}_K\}$  is typically referred to as *situation awareness*. Some immediate questions which arise here, are the accuracy of the sets  $\hat{G}_k$  and more importantly, *when* this information is available.

The probability that target  $i$  survives, despite the allocation against it and destroys the target to which it is aimed, is given by

$$P(\text{target } i \text{ is successful}) = \pi_i \prod_{j=1}^M (1 - p_{i,j})^{x_{i,j}}. \quad (54)$$

Assuming all target-resource allocations are statistically independent, we see that the probability that asset  $k$  survives its attack, is

$$P(\text{asset } k \text{ survives}) = \prod_{i \in \hat{G}_k} \left\{ 1 - \pi_i \prod_{j=1}^M (1 - p_{(i,j)})^{x_{(i,j)}} \right\}. \quad (55)$$

Our objective function, representing the collective average survival of the asset collection, which we seek to maximise, reads:

$$F = \sum_{k=1}^K W_k \prod_{i \in \hat{G}_k} \left\{ 1 - \pi_i \prod_{j=1}^M (1 - p_{(i,j)})^{x_{(i,j)}} \right\} \quad (56)$$

It follows that the constrained optimisation we must solve is:  
Compute

$$F^* \triangleq \max_{\mathbf{x} \in \mathbb{B}} \{F\} = \max_{\mathbf{x} \in \mathbb{B}} \left\{ \sum_{k=1}^K W_k \prod_{i \in \hat{G}_k} \left\{ 1 - \pi_i \prod_{j=1}^M (1 - p_{(i,j)})^{x_{(i,j)}} \right\} \right\} \quad (57)$$

Subject to

$$\Psi(j) = \sum_{i=1}^N x_{i,j} = 1, \quad j = 1, 2, \dots, M. \quad (58)$$

**REMARK 9** The problem given by (57) and (58) presents numerous difficulties. Firstly, the objective function (56), in its corresponding relaxed form, is neither convex, nor concave. Further, suppose one imposes an additional constraint  $\Gamma_i \triangleq \sum_{j=1}^M x_{(i,j)} = 1$ . This constraint does not lead to a simplification as previously in the case of the target-value based problem, as the nonlinearities arising from the product terms in (56) are not removed by this second constraint.

**REMARK 10** The static asset-based problem admits a special case of the target-value base problem. Suppose one is defending a single asset, then it is immediate that the problem defined at (57) and (58) reduces to the problem defined at (13) and (14).

A result similar to that of Theorem 1, can also be computed for the asset-based allocation problem. Suppose that the target-elimination probabilities  $p_{(i,j)}$ , do not depend upon the index  $j$ , that is, the target-elimination probability only depends upon the target to which it is aimed.

Write  $m_i$ , for the number of defensive resources allocated to target  $i$ . The optimisation problem we now wish to solve has the form:

$$\text{Compute} \quad \max_{m_i \in \mathbb{N}} \left\{ \sum_{k=1}^K W_k \prod_{i \in \hat{G}_k} (1 - \pi_i (1 - p_k)^{m_i}) \right\} \quad (59)$$

$$\text{Subject to} \quad \sum_{i=1}^N m_i = M. \quad (60)$$



In the PhD Thesis of Hosein [40], it was shown that the optimal asset-based defence is obtained by setting  $|m_\alpha - m_\beta| \leq 1$ ,  $\forall \alpha, \beta \in \hat{G}_k$ , for  $k = 1, 2, \dots, K$ . This result states that the best allocation, for the given assumptions and the situation awareness  $\{\hat{G}_1, \dots, \hat{G}_K\}$ , is to allocate resources subset-wise (ie for each  $G_k$ ), as uniformly as is possible.

### 3.3 The Complexity of Static Models

A perennial and challenging question in optimisation is this, “what makes one algorithm more difficult than another?”. Surprisingly this question is highly nontrivial, however, the Theory of Computation (see [73]) offers some answers. This Theory proposes, among other things, a classification scheme for ranking the complexity of algorithms, measured by computational time requirements (roughly speaking). The standard rankings are,  $P$  (polynomial time),  $NP$  (non-deterministic polynomial time) and finally  $NPC$  (non-deterministic polynomial time complete). The order or degree of complexity, from least to most, is  $P$ ,  $NP$  then  $NPC$ . Given this system of classification, the immediate question in our context is this, “how might one compute the complexity of a defensive resource allocation algorithm?”. The reason for posing this question is obvious, given the time critical nature of defensive resource allocation.

In the article by Lloyd and Witsenhausen [49], the static target-value based weapon target allocation problem was shown to be  $NPC$ . Usually the complexity of a given algorithm is established by one to two approaches, 1) direct means by first principles, or 2) showing the given algorithm is *equivalent* to another algorithm whose classification is already known. The second approach was used in [49], by establishing the equivalence of the static target-value problem to the so called EXACT 3-COVER problem, which is known to be  $NPC$ . The consequences of this result are significant for both the online and offline scenarios in defensive resource allocation. Since one can show that the static target-value based problem is a special case of the static asset-based problem, then it would naturally follow, that asset-based allocation problems are most likely to be  $NPC$ , although this has not been established in the literature.

## 4 Example

For an example we consider the target-value based static allocation problem. In particular we examine the sensitivity of the model given at (10), (34) and (35) against an uncertain matrix of elimination probabilities. The precision of elimination probabilities is at best speculative, however, without such numbers, one would have essentially very little to work with. In this sense, target-elimination probabilities are not unlike nominal model transition probabilities in jump Markov systems, that is, one must simply accept these quantities as given.

The scenario we consider, is 5 ranked threats (targets) and 5 defensive resources. The

threats values and target-elimination probabilities, are respectively,

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0.1122 \\ 0.4433 \\ 0.4668 \\ 0.0147 \\ 0.6641 \end{bmatrix}, \quad [p_{(i,j)}] = \begin{bmatrix} 0.7000 & 0.1000 & 0.2300 & 0.5000 & 0.4000 \\ 0.2000 & 0.7000 & 0.2500 & 0.1000 & 0.5000 \\ 0.5000 & 0.2000 & 0.7000 & 0.1000 & 0.2300 \\ 0.4000 & 0.2000 & 0.3000 & 0.7000 & 0.1000 \\ 0.1000 & 0.2000 & 0.3000 & 0.4000 & 0.7000 \end{bmatrix}. \quad (61)$$

The matrix  $[p_{(i,j)}]$  at (61) is taken as the true matrix of target-elimination probabilities. What we wish to do, is replace this matrix with a design matrix, whose elements are different to those in  $[p_{(i,j)}]$ , then compute the *optimal* allocation, given the design matrix just described. To measure the consequences of using an allocation computed via perturbed probabilities, we use the average survivability of the threat, that is, we evaluate (10), using the true probability matrix at (61), but for the given allocation computed via the perturbed probability matrix. Rather than perturb all elements of the matrix  $[p_{(i,j)}]$ , we choose only those elements with a value 0.7. For design matrices, we replace each of these elements with an incorrect value, from regularly spaced values in the interval  $[0.1, 0.9]$ . Figure 1 shows the result of this simulation. The sub-figure on the left of Figure 1 is the ascending sorted list of objective function values over the entire space of feasible solutions. These plots can be very useful, as they give an indication to how difficult the optimisation problem might be, particularly for gradient based methods. The sub-figure on the right shows average survivability of the collective threat, given allocations computed via incorrect probabilities, as described above. The independent variable on the right sub-figure is the design value of elimination probability replacing 0.7.

An immediate conclusion from this study, is there are indeed consequences for uncertainties in elimination probabilities, even in this simple static allocation problem.

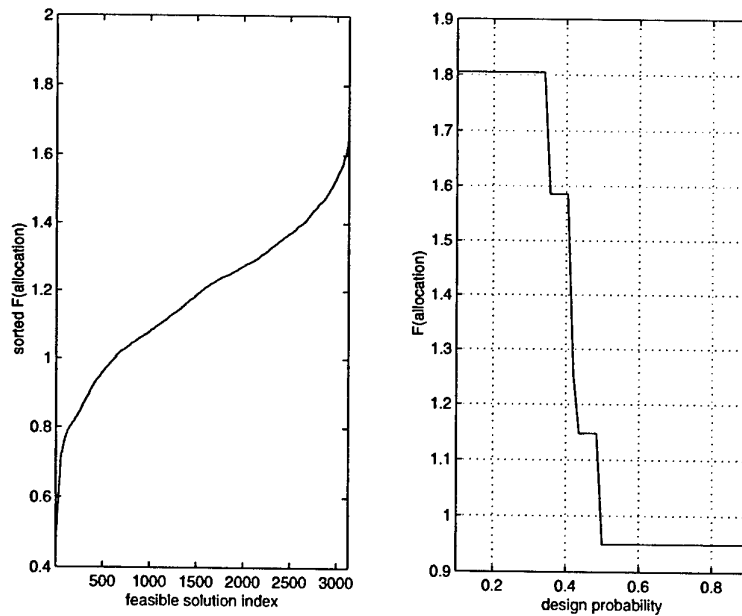


Figure 1: Threat survivability with uncertain target-elimination probabilities.

## 5 Conclusion

### 5.1 Material Presented

Basic time-independent (static) defensive resource allocation problems have been discussed. A brief literature review has been given, identifying major contributions and a diversity of approaches to computing solutions. An important special case of the target-value allocation problem admitting a transportation problem has been discussed in detail. Computer simulations were provided, emphasising the susceptibility of resource allocation problems to parameter uncertainties. Technical details have been kept to a minimum in this report, with more emphasis placed upon the character, complexity and importance of defensive resource allocation.

### 5.2 Customer Relevance

The customer (ADF) relevance of defensive resource allocation research is immediate and needs little justification, however, some distinctions should be stressed. Where *defensive resources* means weapons, or weapons systems, one must note that such resources are prohibitively expensive and complex in their nature. Due to these limitations, research can at best offer reliable summary measures of speculated performance, such as probabilities or confidence intervals. In this sense, it becomes clear that the offline scenario described in the introduction is highly relevant. For DSTO to provide meaningful advice to the ADF on weapon target allocation, such as the summary measures described above, it is critical that an offline simulation capability be developed. Such a capability can potentially provide the ADF with a means to: compute measures of defence effectiveness for predicted engagement scenarios, compute measures of preparedness and to aid in acquisition of defensive resources *etc.*

## 6 Future Research

### Asset-based defensive resource allocation:

A core problem with the *popular* and often abused notion of NCW is; given the significant technical challenges of NCW, how might one construct a measure to demonstrate the return/benefit of NCW ? Currently this question remains unanswered. It is important to note a subtle distinction here, the issue is not primarily how to show the benefits of NCW, rather, how to construct a reliable measure, which *then* may be used to quantify the benefits, or otherwise, of NCW.

The asset-based weapon target allocation might provide an opportunity to respond to this question and to do so in a meaningful and quantitative manner. Recall that the very basic notion of NCW is information sharing, or roughly, in defence parlance, situation awareness. It is supposed that any *single* node in a given network will enjoy situation awareness information enhanced by *all* nodes in the network. In asset-based weapon target allocation, one must have the sets  $\hat{G}_k$ , described above in section 3.2. Now suppose one considers the question of scheduling RADAR. Scheduling RADAR is a stochastic control

problem, where the control variate chooses a sensor according to meeting some objective. In some cases a scheduling control process might be driven by the minimisation of a mean square error in a track estimate. Suppose, however, RADAR is scheduled by a control variable, whose objective is to maximise the particular situation awareness needed by asset-based weapon target allocation objective functions. This research question has been proposed, (by the author), to Prof. Robyn Evans of Melbourne University in February 2003, and is now being pursued through the Center for Enhancing Network Decision of Sensors and Systems (CENDSS). It is hoped that this research might provide a meaningful and much needed figure of merit for NCW. The intention is to show how enhanced situation awareness, provided by a networked defence, might increase the effectiveness of defensive resource allocation.

#### Soft-kill defensive resource allocation:

Throughout the defensive resource allocation literature, it is assumed that the target elimination probabilities, viz,  $p_{(i,j)}^{HK}$ , correspond to the likelihood of success of the so called "Hard Kill" devices, for example missiles. However, "Soft Kill" devices, such as jammers and decoys, also have an elimination probability  $p_{(i,j)}^{SK}$ . It is not clear how one might define a resource allocation problem including both  $p_{(i,j)}^{HK}$  and  $p_{(i,j)}^{SK}$ , as the impact upon a given threat of hard kill and soft kill devices, are not necessarily the same. However, given the significant benefits of soft kill devices, i.e. defence effectiveness and cost effectiveness, then it is natural to ask how defensive resource allocation models should be developed incorporating a general class of resources, that is, both hard and soft kill resources.

#### Stochastic integer programming:

It was shown in this report, that a particular weapon target allocation problem could be cast as a linear integer program. More generally, weapon target allocation problems are nonlinear, however, integer programming can still be applied to nonlinear constrained optimisation problems. Some recent results in stochastic integer programming potentially offer a means of solving defensive resource allocation problems, in particular, stochastic versions of the standard techniques, see [47,71].

#### Neuro-dynamic programming / Reinforcement learning:

It is clear from the recent article of Bertsekas *et al* [6], that neuro-dynamic programming potentially offers a promising approach to defensive resource allocation. However, it would seem that online applications using NDP are not yet feasible. Nonetheless, this approach offers a means to analyse complex problems in the offline scenario. Further, interest and funding in NDP is growing rapidly.

#### Robustness:

A common feature of every defensive resource allocation scheme, static, dynamic, target-value based, asset-value based, is that parameter values are often assumed known. In practice this assumption will never hold. Further, the simulation study given in section 4 shows that there are significant consequences arising from parameter uncertainty. To address this problem, one must essentially develop a new model, (objective function), explicitly including a model for risk or uncertainty. Some recent results ([61]), have

considered incorporating notions of risk arising in quantitative finance into weapon target allocation problems. Clearly defensive resource allocation is a decision making problem with a significant component of risk, yet this risk is currently ignored by all standard models. There is, therefore, considerable scope to include explicit modelling of risk in defensive resource allocation.

In the last ten years quantitative finance has experienced a profound rate of growth. In particular, the rich intersection between quantitative finance and modern Electrical Engineering is being pursued with intensity. Notions of risk are central to quantitative finance. Further, despite a common misconception that quantitative finance strives primarily to predict the behavior of markets, in general, its real objective is to quantify and manage risk. It's now clear that there exists a substantial potential to adapt extant finance-based models for risk, such as conditional value at risk (CVAR), into the problems of defensive resource allocation. Appreciating the intrinsic sensitivity of integer programming, upon which many weapon target allocation algorithms are based, one should immediately realise that quantitative modelling of risk in defensive resource allocation is a vital area of future research.

#### Parallel processing:

Finally, it is clear, that even the 'best' algorithms for defensive resource allocation, (considered in an offline scenario), may still require considerable, if not prohibitive computer processing time. This is not surprising, given the complexity characteristics briefly discussed in section 5. It was suggested by Hosein (1989) [40], that algorithms for weapon target allocation are amenable to parallel processing and therefore should be considered. This observation remains just as relevant today [7,11].

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## Appendix A Linear Programming

Consider the system of linear equations

$$A\mathbf{x} = \mathbf{b}. \quad (\text{A1})$$

Here  $\mathbf{x} \in \mathbb{R}^{n \times 1}$ ,  $\mathbf{b} \in \mathbb{R}^{m \times 1}$  and  $A \in \mathbb{R}^{m \times n}$ . Assume  $A$  is of rank  $m$ . We compose a matrix  $B$  from the  $m$  linearly independent columns of  $A$ . Since  $B$  is nonsingular the system of equations

$$B\mathbf{x}_B = \mathbf{b} \quad (\text{A2})$$

has a unique solution.

Write

$$\mathbf{x} \triangleq (\mathbf{x}'_B, \mathbf{0}')' \in \mathbb{R}^{n \times 1}. \quad (\text{A3})$$

The  $\mathbf{x}$  defined above motivates the following definition.

**DEFINITION 3 (BASIC SOLUTIONS)** *Consider the system of equations at (A1), with rank equal to  $m$  and let  $B \in \mathbb{R}^{m \times m}$ , be any sub-matrix composed by  $m$  columns of  $A$ . If all  $n - m$  components of  $\mathbf{x}$  not associated with columns of  $B$  are assigned the value zero, the solution of the resulting system of equations is said to be a basic solution to (A1) with respect to the basis  $B$*

The following Theorem is standard, see [50], although its importance cannot go understated, as it essentially forms the foundation of the Simplex algorithm. It is included here for completeness.

**THEOREM 2 (FUNDAMENTAL THEOREM OF LINEAR PROGRAMMING)** *Consider a given linear program in standard form:*

$$\text{minimise} \quad \langle \mathbf{c}, \mathbf{x} \rangle, \quad (\text{A4})$$

$$\text{subject to,} \quad A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}. \quad (\text{A5})$$

Here the matrix  $A \in \mathbb{R}^{m \times n}$  is of rank  $m$ . For the stated linear program, the following two conditions hold:

1. If there is a feasible solution, then there is a basic feasible solution.
2. If there is an optimal feasible solution, then there is an optimal basic feasible solution.

### Proof of Theorem 2

1. Suppose  $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$  is a feasible solution, then

$$\mathbf{b} = \mathbf{x}_1 \mathbf{a}_1 + \mathbf{x}_2 \mathbf{a}_2 + \dots + \mathbf{x}_n \mathbf{a}_n. \quad (\text{A6})$$

Here  $\mathbf{a}_i$  denotes the  $i$ -th column of the matrix  $A$ . Given the constraint  $\mathbf{x} \geq \mathbf{0}$ , some of the  $\mathbf{x}$  components in this solution might be zero. Assume that  $p$  of these components are not zero and for convenience, write

$$\mathbf{b} = x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \cdots x_p \mathbf{a}_p. \quad (\text{A7})$$

It is not clear whether the set  $\{\mathbf{a}_1, \dots, \mathbf{a}_p\}$ , is linearly independent, or linearly dependent. This uncertainty admits two cases.

*Case 1 (Linear Independence):* Clearly  $p \leq m$ . If  $p = m$ , then the solution is basic and we are done. Suppose  $p < m$ . Since  $A$  has rank  $m$ , then  $m - p$  vectors can be found from the remaining set of  $m$  linearly independent column vectors. Setting the value zero to the corresponding  $m - p$  variables leads to a basic feasible solution.

*Case 2 (Linear Dependence):* If the column vectors  $\{\mathbf{a}_1, \dots, \mathbf{a}_p\}$  are linearly dependent, then there exists a nontrivial linear combination of these vectors that is the zero vector, that is

$$\mathbf{0} = y_1 \mathbf{a}_1 + y_2 \mathbf{a}_2 + \cdots y_p \mathbf{a}_p, \quad (\text{A8})$$

for constants  $y_1, \dots, y_p$ , at least one of which may be assumed to be positive. Now multiply the equation above by the positive scalar  $\epsilon$ , then subtract the result from (A7), to form

$$\mathbf{b} = (x_1 - \epsilon y_1) \mathbf{a}_1 + (x_1 - \epsilon y_2) \mathbf{a}_2 + \cdots (x_1 - \epsilon y_p) \mathbf{a}_p. \quad (\text{A9})$$

Equation (A9) will hold for each  $\epsilon$ . The set of components  $\{(x_1 - \epsilon y_1), \dots, (x_1 - \epsilon y_p)\}$  corresponds to a solution of the linear equalities  $A\mathbf{x} = \mathbf{b}$ , however, note that weak inequality  $x_i - \epsilon y_i \geq 0$  may be violated. Write

$$\mathbf{y} \triangleq (y_1, y_2, \dots, y_p, 0, \dots, 0)' \in \mathbb{R}^{n \times 1}. \quad (\text{A10})$$

For each  $\epsilon$ , the vector  $\mathbf{x} - \epsilon \mathbf{y}$  satisfies  $A\mathbf{x} = \mathbf{b}$ . Suppose now that we increase  $\epsilon$  from zero, in doing so the components of (A9) will increase, decrease, or remain constant, depending upon  $y_i$ , being respectively, negative, positive, or zero. Recall that we assumed at least one  $y_i$  is positive, so there exists a component which will decrease as  $\epsilon$  is increased. Consider increasing  $\epsilon$  to a magnitude at which one or more components of (A9) become zero. Write

$$\epsilon^* \triangleq \min\{x_i/y_i \mid y_i > 0\}. \quad (\text{A11})$$

The solution  $\mathbf{x} - \epsilon^* \mathbf{y}$  is a feasible solution and has at most  $p - 1$  positive variables. Clearly we can keep repeating this process, choosing further  $\epsilon^*$  as before and in so doing, arrive at a feasible solution formed by only linearly independent column vector of  $A$ , then we consider Case 1.

2. To prove the second claim we can begin by proceeding as before, however, we must establish that  $\mathbf{x} - \epsilon \mathbf{y}$  is the optimal solution. If we suppose  $\mathbf{x} = (x_1, x_2, \dots, x_n)'$  is an optimal solution and that there are exactly  $p \leq n$  positive components in this vector, then we can work through the two cases of linear independence and linear dependence as before. This is omitted.

Now, recall that our objective, is to minimise  $\langle \mathbf{c}, \mathbf{x} \rangle$ , subject to  $A\mathbf{x} \geq \mathbf{0}$ . For the feasible solution  $\mathbf{x} - \epsilon \mathbf{y}$ , we note that

$$\langle \mathbf{c}, \mathbf{x} - \epsilon \mathbf{y} \rangle = \langle \mathbf{c}, \mathbf{x} \rangle - \epsilon \langle \mathbf{c}, \mathbf{y} \rangle. \quad (\text{A12})$$

For an appropriate  $\epsilon$ ,  $\mathbf{x} - \epsilon\mathbf{y}$  is a feasible solution, so we must conclude that the vector  $\mathbf{y}$  is orthogonal to the vector  $\mathbf{c}$ , that is  $\langle \mathbf{c}, \mathbf{y} \rangle = 0$ . Suppose this were not the case. If  $\langle \mathbf{c}, \mathbf{y} \rangle \neq 0$ , then we could choose an even smaller  $\epsilon$ , making (A12) yet smaller, while retaining feasibility. This outcome would violate the assumption that  $\mathbf{x}$  is optimal, so it must be true that  $\langle \mathbf{c}, \mathbf{y} \rangle = 0$ . It follows that claim 2 in Theorem 2 is true.

□

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In this article we consider two classes of static defensive resource allocation problems, these are, the static "target-value based" weapon target allocation and the static "asset-value based" weapon allocation problem. It is shown that the target-value based problem can be recast, (using indicator functions), into an instantiation of the so-called transportation problem. The transportation problem can be solved by numerous polynomial-time algorithms and has received considerable attention in the literature. We also consider the so-called "asset-based" weapon target allocation problem. This problem is shown to be somewhat more difficult than the target value based problem. A simulation study is presented for the target-value allocation problem, with emphasis upon sensitivity to uncertain target-elimination probabilities.